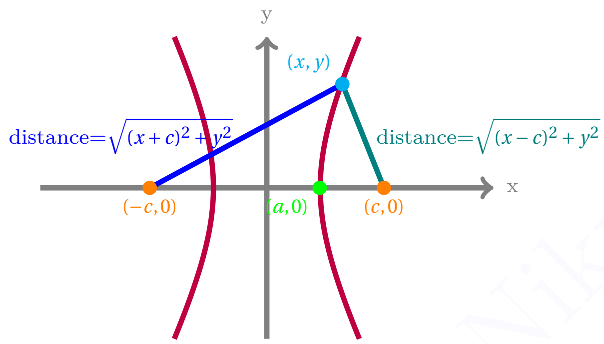
10.2: Hyperbolas

- Geometric definition: The set of points whose **difference** in distances from two points (called **foci**) is constant.
- Using the geometric definition to find a formula



Note that in the picture the difference in distances between the vertex $(a, 0)$ and each of the foci is $2a$ so the differences in the distances between any point on the hyperbola and each of the foci should be equal to $2a$.

$$\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} = 2a$$

Manipulate the same way you would when solving equations with two radicals. Solving for x or y renders the same answer:

Isolate one radical: $\Longrightarrow$ $$\sqrt{(x-c)^2+y^2} = 2a + \sqrt{(x+c)^2+y^2}$$

Raise to Power 2: $\Longrightarrow$ $$(x-c)^2 + y^2 = 4a^2 + 4a\sqrt{(x+c)^2+y^2} + (x+c)^2 + y^2$$

Use binomial expansion and simplify: $\Longrightarrow$ $$0 = 4a^2 + 4a\sqrt{(x+c)^2+y^2} + 4xc$$

factor 4 and isolate the radical: $\Longrightarrow$ $$-a\sqrt{(x+c)^2+y^2} = a^2 + cx$$

Raise to power 2 again: $\Longrightarrow$ $$a^2(x+c)^2 + a^2y^2 = a^4 + 2a^2cx + c^2x^2$$

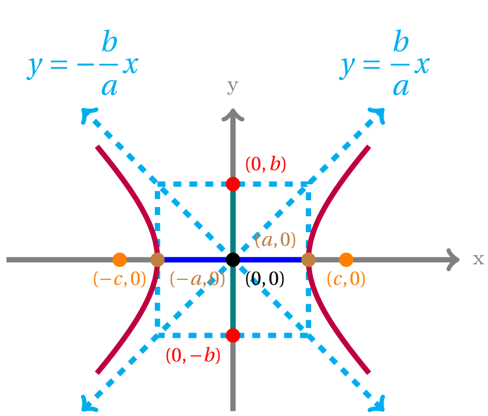
Simplify: $\Longrightarrow$ $$a^2x^2 + a^2c^2 + a^2y^2 = a^4 + c^2x^2$$

Isolate the terms with variables: $\Longrightarrow$ $$(a^2-c^2)x^2 + a^2y^2 = a^4 - a^2c^2$$

Denote $a^2 - c^2$ by $-b^2$: $\Longrightarrow$ $$-b^2x^2 + a^2y^2 = a^2b^2$$

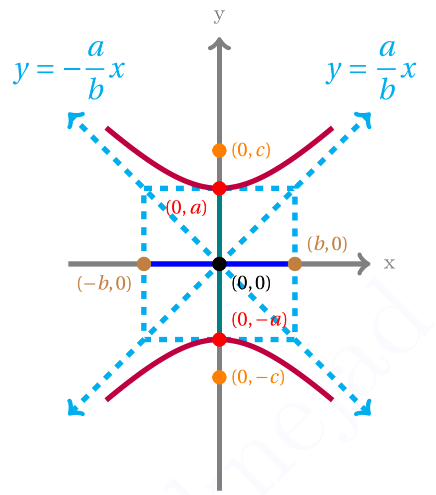
Divide by $-a^2b^2$: $\Longrightarrow$ $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

- Graphs of hyperbolas where axes are vertical or horizontal



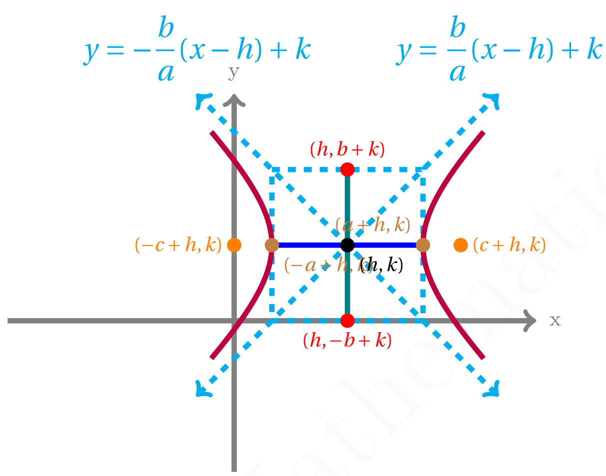
Horizontal hyperbola

Standard Equation: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



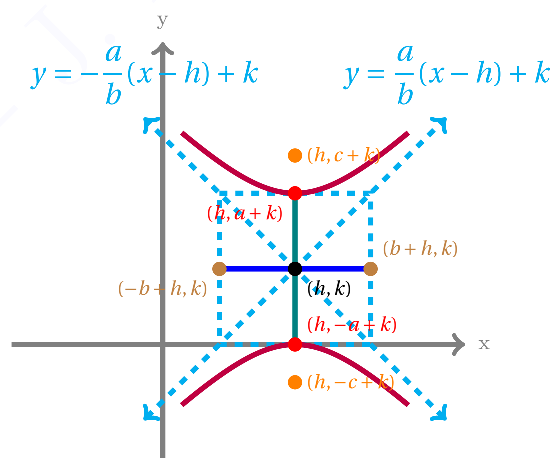
Vertical Hyperbola

Standard Equation: $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$



Horizontal hyperbola and center (h, k)

Standard Equation: $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$



Vertical Hyperbola and center (h, k)

Standard Equation: $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$

- How to find different parameters for a hyperbola using its equation:

  1. If the equation is anything other than the above equations, reformat to one of the above.
  2. In standard form if x term is positive, then the hyperbola is horizontal. Otherwise, the hyperbola is vertical.
  3. If hyperbola is horizontal, to find the vertices, plug in $y = 0$ or $y = k$ and solve. If hyperbola is vertical, to find the vertices, plug in $x = 0$ or $x = h$ and solve.
  4. Notice the asymptotes when drawing the parabolas.
  5. Find c using the equation $c^2 = a^2 + b^2$.

1. Write each of the following hyperbola equations in standard form and find a and b. Then find c.

   (A) $25x^2 - 4y^2 = 100$ (B) $4(y-3)^2 - 36(x-5)^2 = 36$ (C) $25y^2 - 4x^2 = 1$

2. Use completing the square to find the standard equation of the following hyperbola, then find the center of the hyperbola. $4x^2 - 24x - 9y^2 + 18y - 9 = 0$

3. Which of the following is an equation for a hyperbola with foci $(0,5)$ and $(0,-5)$ and asymptotes $y = \pm\frac{3x}{4}$?

(a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

(b) $\frac{y^2}{16} - \frac{x^2}{9} = 1$

(c) $\frac{x^2}{9} - \frac{y^2}{16} = 1$

(d) $\frac{y^2}{9} - \frac{x^2}{16} = 1$

4. Sketch the graph of the hyperbola $\frac{x^2}{64} - \frac{y^2}{16} = 1$. Label the vertices, foci and asymptotes.

Videos:

1. Example 1: https://mediahub.ku.edu/media/t/1_4rni0x07
2. Example 2: https://mediahub.ku.edu/media/t/1_txzidlil